Poynting's Theorem

The Poynting theorem is one on the most important in EM theory. It tells us the power flowing in an electromagnetic field.

Poynting vector: describes the direction and magnitude of electromagnetic energy flow and is used in the Poynting theorem

Derivation of Poynting'sTheorem

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \xrightarrow{H\cdot} \underline{H}\cdot\left(\nabla \times \underline{E}\right) = -\underline{H}\cdot\frac{\partial \underline{B}}{\partial t} \qquad (1)$$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \xrightarrow{E\cdot} \underline{E}\cdot\left(\nabla \times \underline{H}\right) = \underline{E}\cdot\underline{J} + \underline{E}\cdot\frac{\partial \underline{D}}{\partial t} \qquad (2)$$

$$\underline{H}\cdot\left(\nabla\times\underline{E}\right)=-\underline{H}\cdot\frac{\partial\underline{B}}{\partial t}\qquad(1)$$

$$\underline{E}\cdot\left(\nabla\times\underline{H}\right)=\underline{E}\cdot\underline{J}+\underline{E}\cdot\frac{\partial\underline{D}}{\partial t}\qquad(2)$$

Subtract, and use the following vector identity

$$\underline{H}.(\nabla\times\underline{E})-\underline{E}.(\nabla\times\underline{H})=\nabla.(\underline{E}\times\underline{H})$$

We will have

$$\nabla.(\underline{E}\times\underline{H})=-\underline{H}\cdot\frac{\partial\underline{B}}{\partial t}-\underline{E}\cdot\underline{J}-\underline{E}\cdot\frac{\partial\underline{D}}{\partial t}$$

$$\nabla.(\underline{E}\times\underline{H}) = -\underline{H}\cdot\frac{\partial\underline{B}}{\partial t} - \underline{E}\cdot\underline{J} - \underline{E}\cdot\frac{\partial\underline{D}}{\partial t}$$

We can substitute by following:

$$-\underline{H}\cdot\frac{\partial\underline{B}}{\partial t} = -\mu(\underline{H}\cdot\frac{\partial\underline{H}}{\partial t}) = -\frac{1}{2}\mu\frac{\partial}{\partial t}(\underline{H}\cdot\underline{H}) = -\frac{1}{2}\mu\frac{\partial}{\partial t}|H|^2 \qquad (d(\frac{1}{2}u^2) = u.du)$$

$$-\underline{E}\cdot\underline{J} = -\sigma|E|^2$$

Assume there is no sources within the volume, so this is related to ohmic power dissipated Within the volume.

$$-\underline{E}\cdot\frac{\partial\underline{D}}{\partial t} = -\varepsilon(\underline{E}\cdot\frac{\partial\underline{E}}{\partial t}) = -\frac{1}{2}\varepsilon\frac{\partial}{\partial t}(\underline{E}\cdot\underline{E}) = -\frac{1}{2}\varepsilon\frac{\partial}{\partial t}(|E|^2)$$

$$\boxed{\nabla.(\underline{E}\times\underline{H}) = -\frac{1}{2}\mu\frac{\partial}{\partial t}|H|^2 - \sigma|E|^2 - \frac{1}{2}\varepsilon\frac{\partial}{\partial t}|E|^2}$$

$$\nabla .(\underline{E}\times\underline{H}) = -\frac{1}{2}\mu\frac{\partial}{\partial t}|H|^2 - \sigma|E|^2 - \frac{1}{2}\varepsilon\frac{\partial}{\partial t}|E|^2$$

Integrating both sides over a volume and then apply the divergence theorem:

$$\int_v \nabla\cdot(\underline{E}\times\underline{H})dv = -\int_v \frac{1}{2}\mu\frac{\partial}{\partial t}|H|^2 dv - \int_v \sigma|E|^2 dv - \int_v \frac{1}{2}\varepsilon\frac{\partial}{\partial t}|E|^2 dv$$

$$\oint_s (\underline{E}\times\underline{H}).\hat{n}\, ds = -\int_v \frac{1}{2}\mu\frac{\partial}{\partial t}|H|^2 dv - \int_v \sigma|E|^2 dv - \int_v \frac{1}{2}\varepsilon\frac{\partial}{\partial t}|E|^2 dv$$

Total power flowing Out of a volume

Instantaneous ohmic power dissipated within the volume

Rate of change in the total energy Stored in the electric field

Rate of change in the total energy Stored in the magnetic field

Power *leaving* the volume *V* enclosed by *S*= *Decreasing electric and magnetic power*- *Dissipating ohmic power*

$$\oint_s (\underline{E} \times \underline{H}).\hat{n}\, ds = -\int_v \frac{1}{2}\mu \frac{\partial}{\partial t}|H|^2 dv - \int_v \sigma |E|^2 dv - \int_v \frac{1}{2}\varepsilon \frac{\partial}{\partial t}|E|^2 dv$$

$$\underline{S} = \underline{E} \times \underline{H} \quad (W/m^2)$$

Poynting's vector: instantaneous "power density" vector associated with an electromagnetic field(perpendicular to electric and magnetic fields)

$$\oint_s \underline{S}.\hat{n} ds = \text{power leaving enclosed volume}$$

Power leaving the volume V enclosed by S= Decreasing electric and magnetic power-Dissipating ohmic power

$$-\oint_s \underline{S}.\hat{n} ds = \text{power flowing into enclosed volum}$$

Power flowing into a closed surface = Increasing stored electric and magnetic + ohmic power density *time varying field* ***it is often desirable to find the average power density which is obtained by integrating the instantaneous Poynting vector over one period and dividing by the period***

uniform plane wave, for propagation in the $+z$ direction was associated with an E_x and H_y component,

$$E_x \mathbf{a_x} \times H_y \mathbf{a_y} = \mathcal{P}_z \mathbf{a_z}$$

In a perfect dielectric, these **E** and **H** fields are given by

$$E_x = E_{x0} \cos(\omega t - \beta z)$$

$$H_y = \frac{E_{x0}}{\eta} \cos(\omega t - \beta z)$$

and thus

$$S_z = \frac{E_{x0}^2}{\eta} \cos^2(\omega t - \beta z)$$

To find the time-average power density, we integrate over one cycle and divide by the period $T = 1/f$,

$$S_{z,av} = \frac{1}{T} \int_0^T \frac{E_{x0}^2}{\eta} \cos^2(\omega t - \beta z)\, dt$$

$$= \frac{1}{2T} \frac{E_{x0}^2}{\eta} \int_0^T [1 + \cos(2\omega t - 2\beta z)]\, dt$$

$$= \frac{1}{2T} \frac{E_{x0}^2}{\eta} \left[t + \frac{1}{2\omega} \sin(2\omega t - 2\beta z) \right]_0^T$$

$$S_{z,av} = \frac{1}{2}\frac{E_{x0}^2}{\eta} \text{ W/m}^2 \tag{56}$$

If we were using root-mean-square values instead of peak amplitudes, then the factor 1/2 would not be present.

Finally, the average power flowing through any area S normal to the z axis is[5]

$$S_{z,av} = \frac{1}{2}\frac{E_{x0}^2}{\eta}A \text{ W}$$

In the case of a lossy dielectric, E_x and H_y are not in time phase. We have

$$E_x = E_{x0}e^{-\alpha z}\cos(\omega t - \beta z)$$

If we let

$$\eta = |\eta|\angle\theta_\eta$$

then we may write the magnetic field intensity as

$$H_y = \frac{E_{x0}}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)$$

Thus,

$$S_z = E_x H_y = \frac{E_{x0}^2}{|\eta|}e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)$$

Now is the time to use the identity $\cos A\cos B \equiv 1/2\cos(A+B) + 1/2\cos(A-B)$, improving the form of the last equation considerably,

$$S_z = \frac{1}{2}\frac{E_{x0}^2}{|\eta|}e^{-2\alpha z}[\cos(2\omega t - 2\beta z - 2\theta_\eta) + \cos\theta_\eta]$$

$$S_{z,av} = \frac{1}{2}\frac{E_{x0}^2}{|\eta|}e^{-2\alpha z}\cos\theta_\eta$$

- The last expression can be found by easily using phasor form of electric and magnetic fields

$$S_{z,av} = \frac{1}{2}\mathrm{Re}(\mathbf{E_s} \times \mathbf{H_s^*}) \quad \mathrm{W/m^2}$$

where in the present case

$$\mathbf{E_s} = E_{x0}e^{-j\beta z}\mathbf{a_x}$$

and

$$\mathbf{H_s^*} = \frac{E_{x0}}{\eta^*}e^{+j\beta z}\mathbf{a_y} = \frac{E_{x0}}{|\eta|}e^{j\theta}e^{+j\beta z}\mathbf{a_y}$$

Example:

12.20. If $\mathbf{E}_s = (60/r)\sin\theta\, e^{-j2r}\, \mathbf{a}_\theta$ V/m, and $\mathbf{H}_s = (1/4\pi r)\sin\theta\, e^{-j2r}\, \mathbf{a}_\phi$ A/m in free space, find the average power passing outward through the surface $r = 10^6$, $0 < \theta < \pi/3$, and $0 < \phi < 2\pi$.

$$< \mathbf{S} >= \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{15\sin^2\theta}{2\pi r^2}\,\mathbf{a}_r \text{ W/m}^2$$

Then, the requested power will be

$$\Phi = \int_0^{2\pi}\int_0^{\pi/3} \frac{15\sin^2\theta}{2\pi r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, r^2 \sin\theta d\theta d\phi = 15\int_0^{\pi/3} \sin^3\theta\, d\theta$$
$$= 15\left(-\frac{1}{3}\cos\theta(\sin^2\theta + 2)\right)\Big|_0^{\pi/3} = \frac{25}{8} = \underline{3.13 \text{ W}}$$

Note that the radial distance at the surface, $r = 10^6$ m, makes no difference, since the power density dimishes as $1/r^2$.

$$d\overline{S} = r^2 \sin\theta d\theta d\phi\, \hat{a}_r + r\sin\theta dr d\phi\, \hat{a}_\theta + r d\theta dr\, \hat{a}_\phi$$

Example:

D11.6. At frequencies of 1 MHz the dielectric constant of ice made from pure water has values of 4.15 while the loss tangent is 0.12. If a uniform plane wave with an amplitude of 100 V/m at $z = 0$ is propagating through such ice, find the time-average power density at $z = 0$ and $z = 10$ m for each frequency.

Ans. 27.1 and 25.7 W/m^2.

$f = 1\,MHz \quad \epsilon_r = 4.15 \quad \tan\delta = 0.12 \quad E = 100\ V/m$

$S_{avg} = \frac{1}{2|\eta|} |E|^2 \cos\theta_\eta \rightarrow |E| = E_0 e^{-\alpha z}$ ($e^{-j\beta z}$, $e^{j\omega t}$ → phase)

$\tan\delta = \frac{\sigma}{\omega\epsilon_0\epsilon_r} \rightarrow \sigma = 0.12 \times 2\pi \times 10^6 \times 8.85 \times 10^{-12} \times 4.15 = 2.769 \times 10^{-5}\ S/m$

$\eta = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{4\pi \times 10^{-7}}{\epsilon_0\epsilon_r - j\,\sigma/\omega}} = 183.98 + j11 = 184.3 \angle 3.4^\circ, \ \cos\theta_\eta = 1$

$j\gamma = j\omega\sqrt{\mu\epsilon_c} = \alpha + j\beta = j\,2\pi \times 10^6 \sqrt{4\pi \times 10^{-7}\epsilon_c} = 2.55 \times 10^{-3} + j0.042 \rightarrow \alpha = 2.55 \times 10^{-3}$

$\therefore S_{avg}\big|_{z=0} = \frac{1}{2|\eta|} |E|^2 \cos\theta_\eta = \frac{(100)^2}{2 \times 184.3} = 27.1\ watt/m^2$

$S_{avg}\big|_{z=10m} = \frac{|100 e^{-10\alpha}|^2}{2 \times 184.3} = S\big|_{z=0} \times e^{-20\alpha} = 25.7\ watt/m^2$